Oscillations of relativistic tori

Numerical study

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EQUILIBRIUM

Equilibrium model: analytic theory

- Perfect fluid:

$$T^{\alpha}_{\beta} = (e+p)u^{\alpha}u_{\beta} + p\delta^{\alpha}_{\beta}$$

- Pure rotation:

$$u^{\mu} = A\left(t^{\mu} + \Omega\phi^{\mu}\right), \quad u_{\nu} = -E\left(\delta^{t}_{\nu} - \ell\delta^{\phi}_{\nu}\right).$$

- Conservation laws:

$$\nabla_{\alpha}\left(\rho u^{\alpha}\right) = 0, \quad \nabla_{\alpha}T^{\alpha}_{\beta} = 0.$$

$$(e+p)\boldsymbol{a} + \boldsymbol{\nabla}p = 0, \quad \boldsymbol{a} = \boldsymbol{\nabla}\ln E - \frac{\Omega\boldsymbol{\nabla}\ell}{1-\ell\Omega}.$$

- This can be integrated if $p = p(n)$, $\rho = \rho(n)$ to

$$\boxed{E\Psi h = \mathrm{const}} \quad \Psi \equiv \exp\left[-\int\frac{\Omega d\ell}{1-\ell\Omega}\right].$$

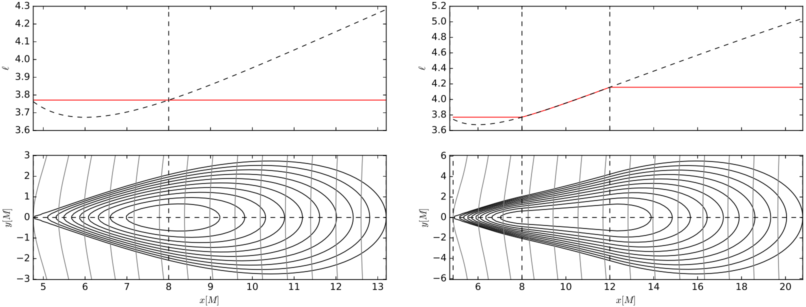
Equilibrium model: construction

Von Zeipel cylindres:

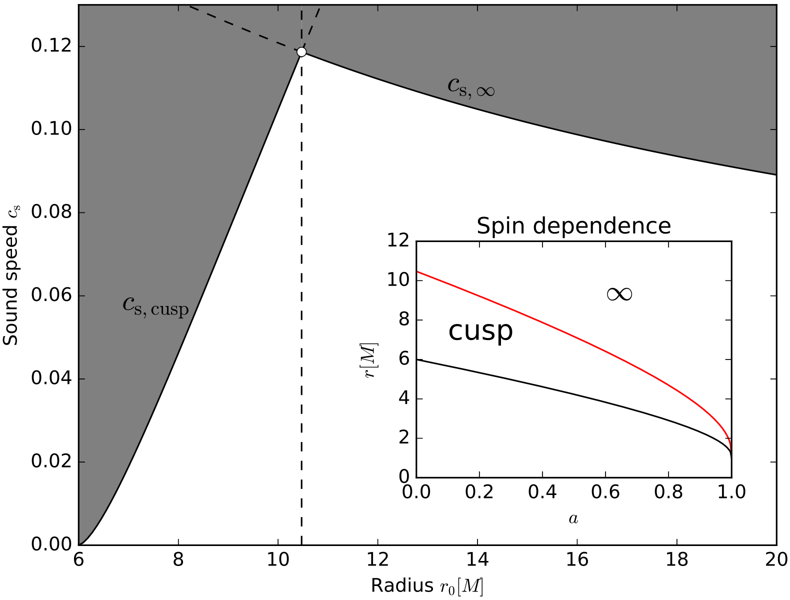
$$\Omega = \Omega(\ell) \quad \text{and} \quad \Omega = \frac{g^{\phi t} - \ell g^{\phi\phi}}{g^{tt} - \ell g^{t\phi}}.$$

Lane-Embden function (polytropes):

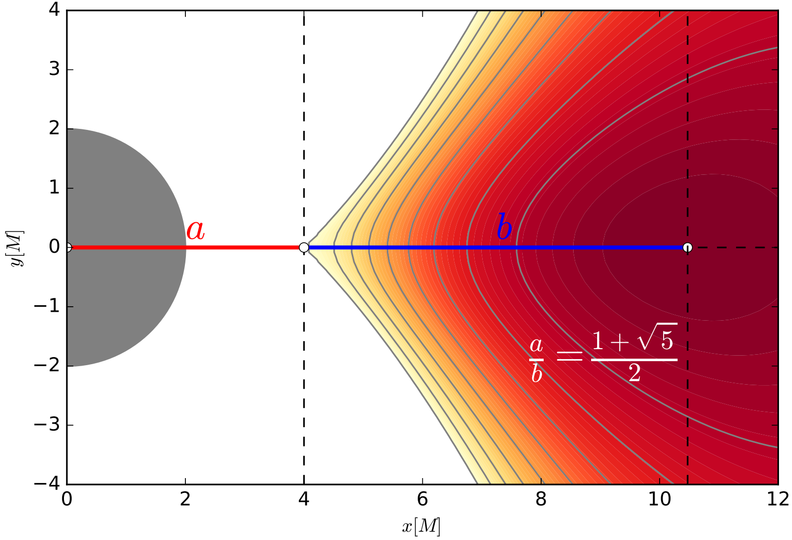
$$f = 1 - \frac{1}{N c_{\mathrm{s0}}^2}\left(1 - \frac{E_0 \Psi_0}{E \psi}\right), \quad 0 \le f \le 1.$$



What limits torus sizes?



Maximal torus in Schwarzschild spacetime



Perturbations

Perturbations

- Perturbation of conservation laws

$$\nabla_\alpha \delta\left(\rho u^\alpha\right) = 0, \quad \nabla_\alpha \delta T^\alpha_\beta = 0.$$

Introduce

$$\tilde{\omega} \equiv \omega - m\Omega, \quad \tilde{m} \equiv m - \ell\omega, \quad \eta \equiv \frac{\delta h}{h}$$

$$\begin{aligned}
-\mathrm{i}\tilde{\omega}\delta\boldsymbol{u} + \boldsymbol{\gamma}_3 \delta u_\phi + \frac{1}{A}\boldsymbol{\nabla}\eta &= 0, \\
-\mathrm{i}\tilde{\omega}\delta u_\phi + \boldsymbol{\gamma}_1 \cdot \delta\boldsymbol{u} + \mathrm{i}\tilde{m}E\eta &= 0, \\
\frac{1}{\rho R}\boldsymbol{\nabla}\cdot(\rho R \delta\boldsymbol{u}) + \frac{\mathrm{i}\tilde{m}E}{AR^2}\delta u_\phi - \frac{\mathrm{i}\tilde{\omega}A}{c_{\mathrm{s}}^2}\eta &= 0.
\end{aligned}$$

- 3 poloidal vectors:

$$\boldsymbol{\gamma}_1 \equiv E^2\boldsymbol{\nabla}\ell, \quad \boldsymbol{\gamma}_2 \equiv \boldsymbol{\nabla}\Omega, \quad \boldsymbol{\gamma}_3 \equiv \boldsymbol{\gamma}_2 - \frac{\boldsymbol{\gamma}_1}{A^2R^2}.$$

Papaloizou-Pringle equation

Perturbation variable

$$W \equiv -\frac{\delta h}{A\tilde{\omega}} = -\frac{\delta p}{A\tilde{\omega}\rho}.$$

The Euler equation gives

$$\delta \boldsymbol{u} = \frac{\mathrm{i}}{h}\left[\mathbf{P}\cdot\boldsymbol{\nabla}W + \boldsymbol{\gamma}_1\frac{\tilde{m}\tilde{\omega}E}{ADR^2}W\right],$$

$$\delta u_\phi = -\frac{1}{hD}\left[\tilde{\omega}\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}W - \tilde{m}AE\left(\tilde{\omega}^2 + \boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_2\right)W\right],$$

with

$$D \equiv \kappa^2 - \tilde{\omega}^2, \quad \kappa^2 \equiv -\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_3, \quad \mathbf{P} \equiv \tilde{\mathbf{g}} + \frac{1}{D}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_3.$$

Papaloizou-Pringle equation

Continuity equation gives

$$\frac{h}{\rho R}\boldsymbol{\nabla}\cdot\left(\frac{\rho R}{h}\mathbf{P}\cdot\boldsymbol{\nabla}W\right)+\left[\frac{A\tilde{\omega}}{\rho R}\boldsymbol{\nabla}\cdot\left(\frac{\rho\tilde{m}E}{DRA^2}\gamma_1\right)+\frac{\tilde{m}^2\tilde{\omega}^2E^2}{DR^2}+\frac{A^2\tilde{\omega}^2}{c_s^2}\right]W=0.$$

For constant angular momentum tori:

$$\boxed{\frac{h}{\rho R}\boldsymbol{\nabla}\cdot\left(\frac{\rho R}{h}\boldsymbol{\nabla}W\right)+\left[\frac{A^2\tilde{\omega}^2}{c_s^2}-\frac{\tilde{m}^2E^2}{R^2}\right]W=0,}$$

$$\tilde{\omega}\equiv\omega-m\Omega,\quad \tilde{m}\equiv m-\ell\omega$$

quadratic eigenvalue problem:

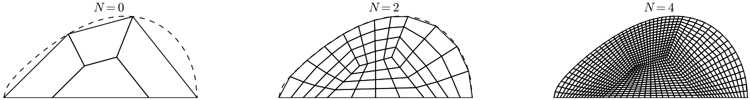
$$\boxed{\hat{L}W+(\omega-m\Omega_1)(\omega-m\Omega_2)W=0,}$$

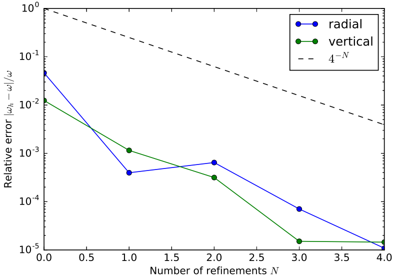
$$\hat{L}\equiv\frac{h}{\rho R\mathcal{B}}\boldsymbol{\nabla}\cdot\left(\frac{\rho R}{h}\boldsymbol{\nabla}\right),\quad \mathcal{B}\equiv\frac{A^2}{c_s^2}-\frac{E^2\ell^2}{R^2},\quad \Omega_{1,2}\equiv\frac{AR\Omega\pm c_sE}{AR\pm c_sE\ell}.$$

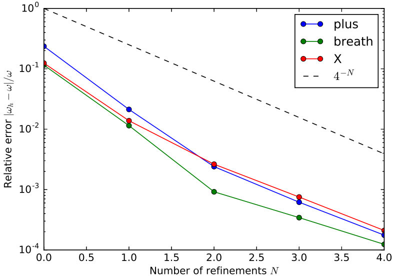
FEM implementation

Diskretized form [$\varphi_b(x)$ = test functions]:

$$\boxed{\sum_b \left(S_{ab} - \omega M_{ab}\right) W_b = 0,} \qquad W(x) = \sum_b W_b \varphi_b(x)$$



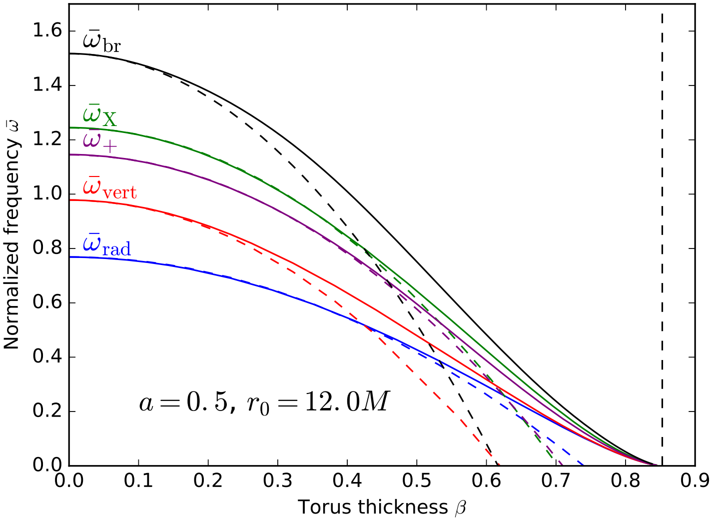




RESULTS

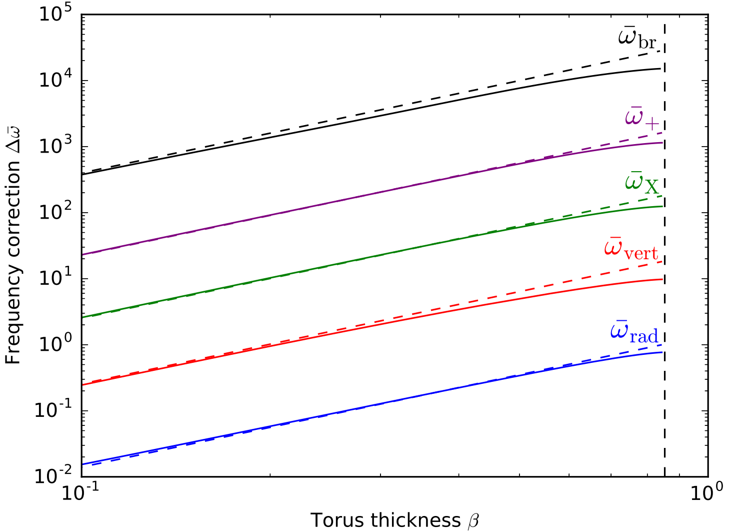
axisymmetric modes

Axisymmetric modes



...Comparison to analytic results of Straub, Šrámková, Blaes

Axisymmetric modes



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RESULTS

non-axisymmetric modes

Cusp tori as model for NS QPOs

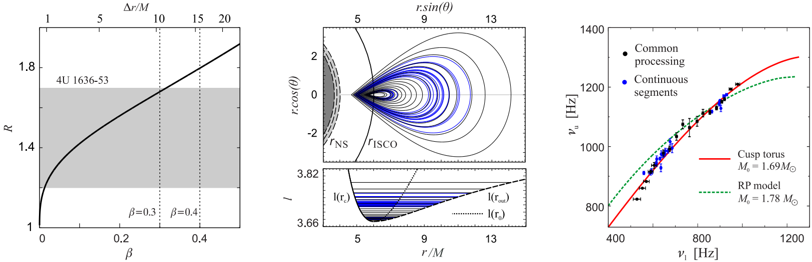
- Identification of the two high-frequency QPOs:

$$\nu_{\rm u} \equiv \Omega(r_0), \quad \nu_{\rm l} \equiv \omega_{{\rm rad},m=1}(r_0,\beta)$$

- Slender torus: $\nu_{\rm l} = \Omega(r_0) - \omega_r(r_0)$
- Sequence of cusp-filling tori: $\beta = \beta_{\rm max}(r_0)$

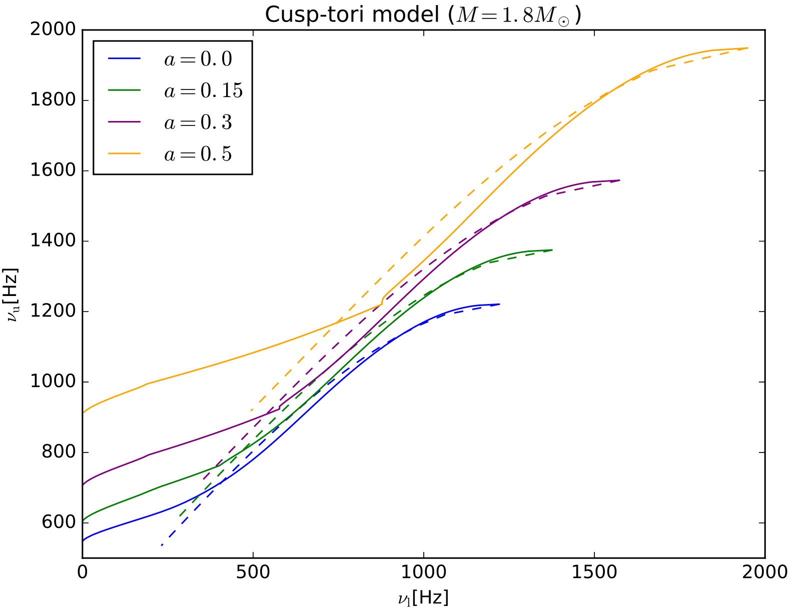
$$\nu_{\rm u} \equiv \Omega(r_0), \quad \nu_{\rm l} \equiv \omega_{{\rm rad},m=1}[r_0,\beta_{\rm max}(r_0)] \equiv \nu_{\rm l}(r_0)$$

$\Rightarrow$ one-parametric relation between $\nu_{\rm u}$ and $\nu_{\rm l}$.

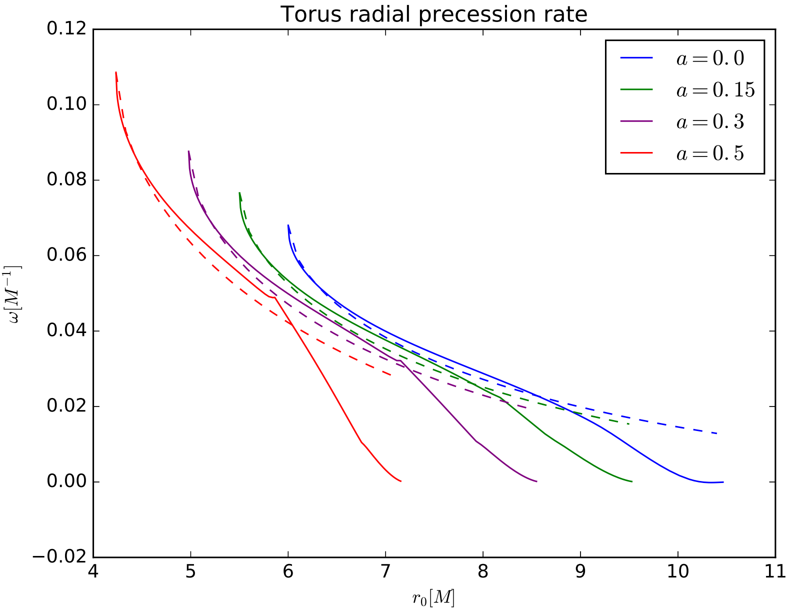


... based on analytic approximations of Straub, Šrámková, Blaes

Cusp tori as model for NS QPOs

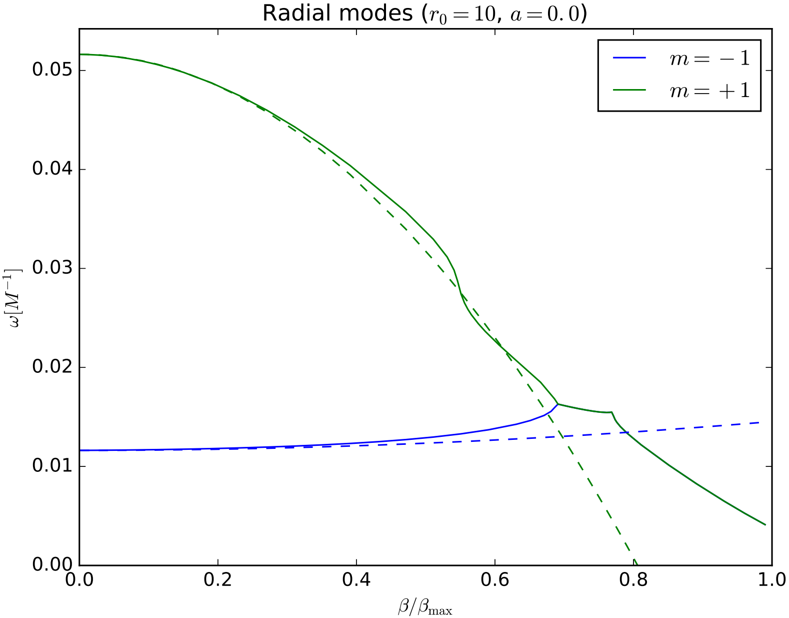


Radial precession ($m = 1$ radial mode)



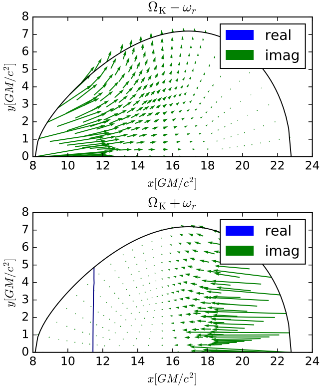
Cusp tori, dashed – analytic predictions of Straub, Šrámková, Blaes

Mode merging ($m = 1$ radial modes)



Cusp tori, dashed – analytic predictions of Straub, Šrámková, Blaes

Mode merging causes instability

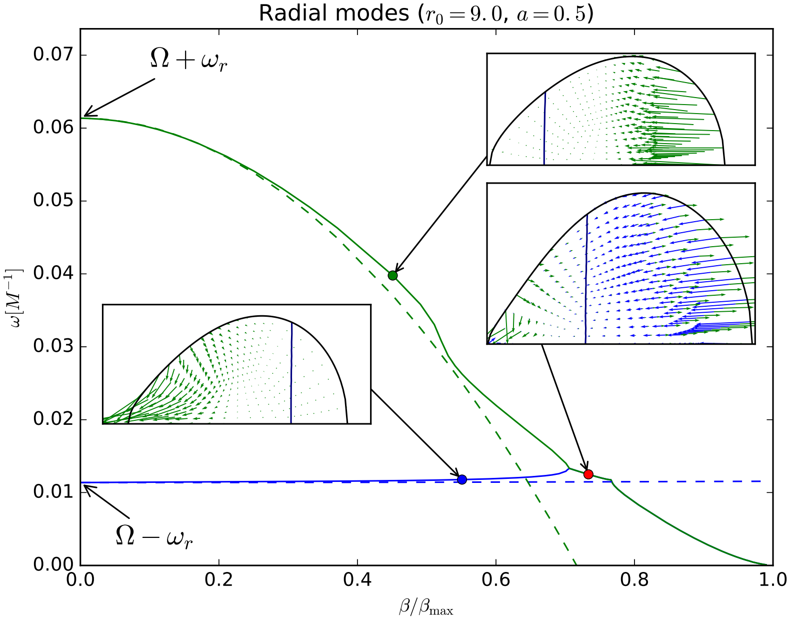


- Inner, $\tilde{\omega} \sim -\omega_r < 0$
- Perturbation **decreases** the total energy of the flow
- **Negative** mode energy

- Outer, $\tilde{\omega} \sim \omega_r > 0$
- Perturbation **increases** flow energy
- **Positive** mode energy

Overreflection $\Rightarrow$ Neutral modes.

Mode merging ($m = 1$ radial modes)



Cusp tori, dashed – analytic predictions of Straub, Šrámková, Blaes