Chaotic motion in the Johannsen-Psaltis spacetime

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Astrophysical setting

- "No-hair" theorem
- Kerr hypothesis

The Kerr spacetime: symmetries $\implies$ integrals of geodesic motion:

- stationarity $\implies$ energy E,
- axisymmetry $\implies$ z-component of angular momentum L_z,
- a "hidden symmetry" $\implies$ Carter constant $\mathcal{K}$.

Thus, geodesic motion in the Kerr spacetime background is integrable.

Non-Kerr spacetimes

Integrability broken by perturbation:

- increasing the number of degrees of freedom,
- reducing the system's degree of symmetry.

"Non-Kerr" spacetime:

- family of spacetimes which parametrically deviate from the Kerr spacetime,
- typical feature of non-Kerr spacetime: geodesic non-integrability.

EMRIs

Detection:

- gravitational wave signals,
- electromagnetic signals,

from EMRIs (Extreme Mass Ratio Inspiral).
Analysis methods:

- frequency analysis,
- recurrence analysis.

Geodesic motion in the Johannsen-Psaltis spacetime

- Metric:

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi \quad (1)$$

- Parameters: M, a, ϵ_2, ϵ_3, ϵ_4...
- Motion generated by the action

$$\mathcal{S} = \int_{\tau_1}^{\tau_2} \mathcal{L}\mathrm{d}\tau, \quad \mathcal{L}\left(x^\mu, \dot{x}^\mu\right) = \frac{m}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \quad (2)$$

- Autonomous system (evolution parameter τ) $\Longrightarrow$ conservation of the Hamiltonian

$$\mathcal{H} = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu \quad (3)$$

- Integrals $\mathcal{H}$,E,L_z (and in the Kerr case $\mathcal{K}$) are independent and in involution ($\{I_i, I_j\} = 0$)

Numerical accuracy

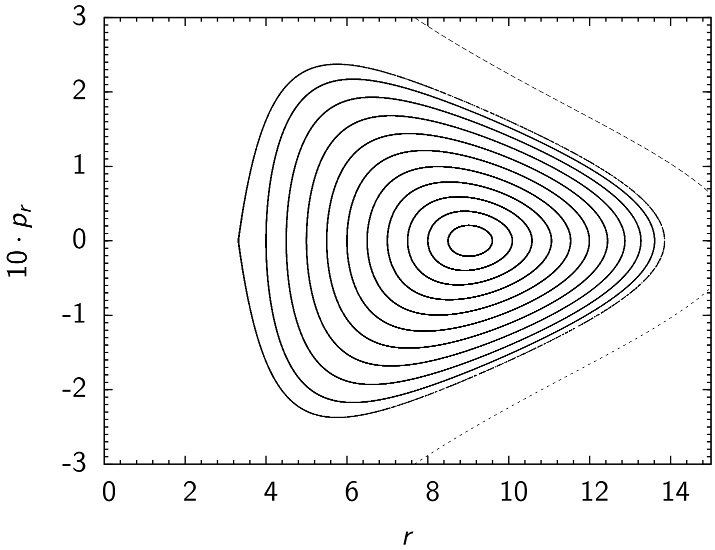
Calculated using:

- program written in C,
- RK4 algorithm,
- accuracy tracked by $\Delta\mathcal{L}_{rel}$.

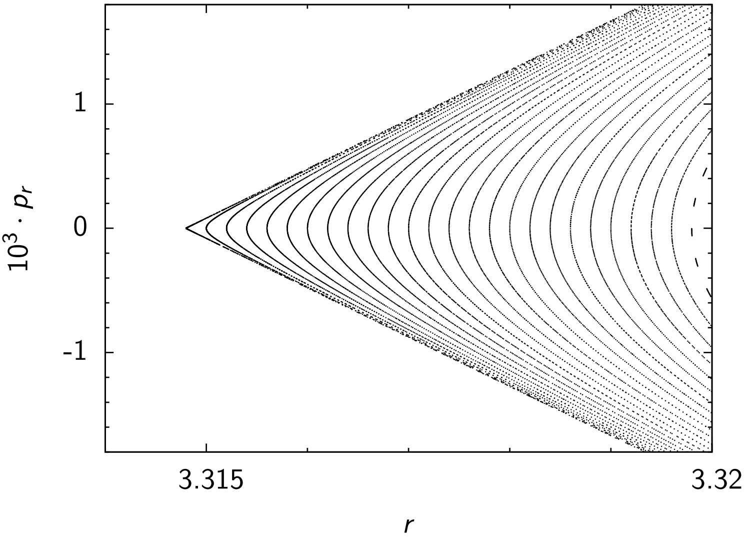
Surface of section

- A plane in the phase space of a system
- Used here: the equatorial plane $\theta = \pi/2$ with $\dot{\theta} \geq 0$
- Liouville-Arnol'd theorem: nested invariant circles in integrable case

Surface of section in an integrable system



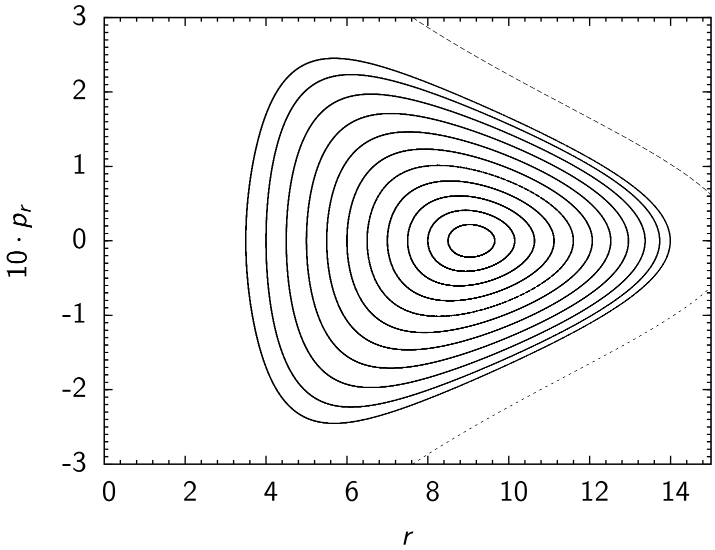
Surface of section in an integrable system



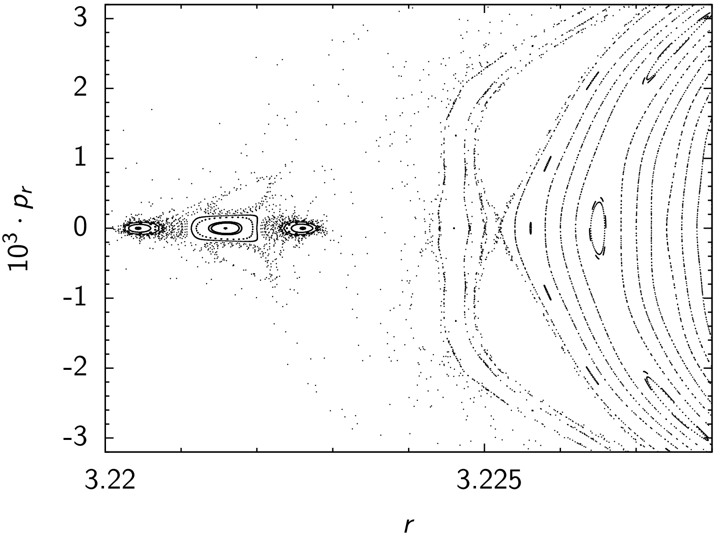
Surface of section in the perturbed system

- KAM theorem: some circles remain after perturbation
- Poincare-Birkhoff theorem: island chains

Surface of section in the perturbed system



Surface of section in the perturbed system



Rotation number

Regular trajectories:

- fundamental frequencies ω^r, ω^θ
- rotation number $\omega = \omega^r / \omega^\theta$

Take angles between consecutive points: $\vartheta_n = \operatorname{ang}\left[x_n, x_{n+1}\right]$

- for regular trajectories

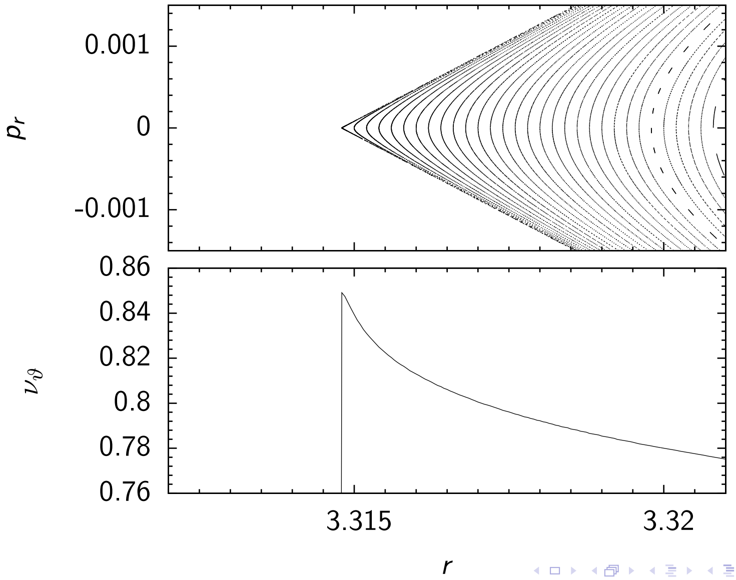
$$\nu_\vartheta = \lim_{N\to\infty} \frac{1}{2\pi N} \sum_{i=1}^{N} \vartheta_i = \omega \qquad (4)$$

- alternative: frequency analysis (Laskar, 1993)

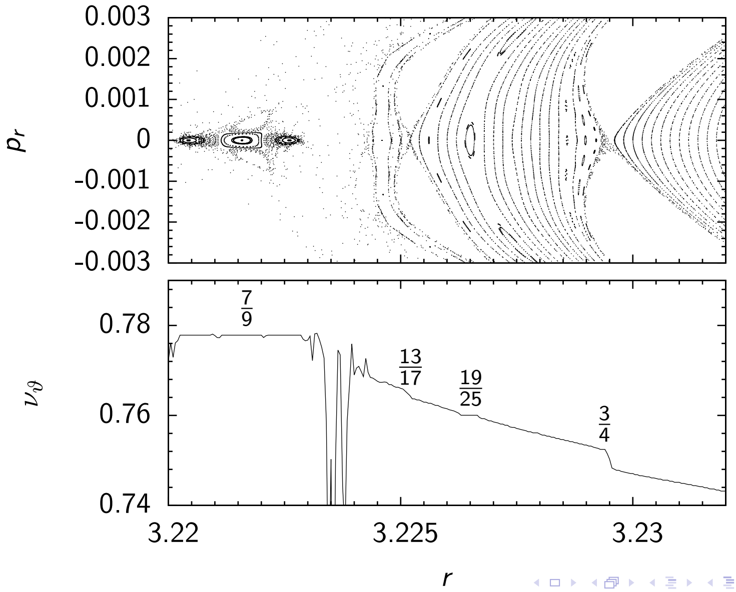
Rotation number

- ν_ϑ can be calculated for a chaotic trajectory as well
- in an integrable system: strictly monotonic function of the initial condition
- non-integrable system: non-monotonic variations when passing through chaotic zones

Rotation number in an integrable system



Rotation number in the perturbed system



Lyapunov exponents

Deviation vector: between two infinitesimally close trajectories

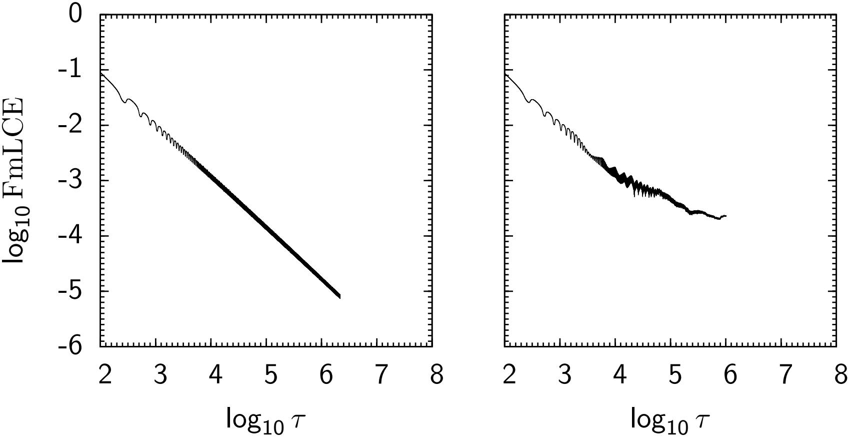
$$\ddot{\xi}^{\mu} + \frac{\partial \Gamma^{\mu}{}_{\kappa\lambda}}{\partial x^{\nu}} \dot{x}^{\kappa} \dot{x}^{\lambda} \xi^{\nu} + 2\Gamma^{\mu}{}_{\kappa\lambda} \dot{x}^{\kappa} \dot{\xi}^{\lambda} = 0 \tag{5}$$

$$\Xi^2 = g_{\mu\nu} \xi^{\mu} \xi^{\nu} \tag{6}$$

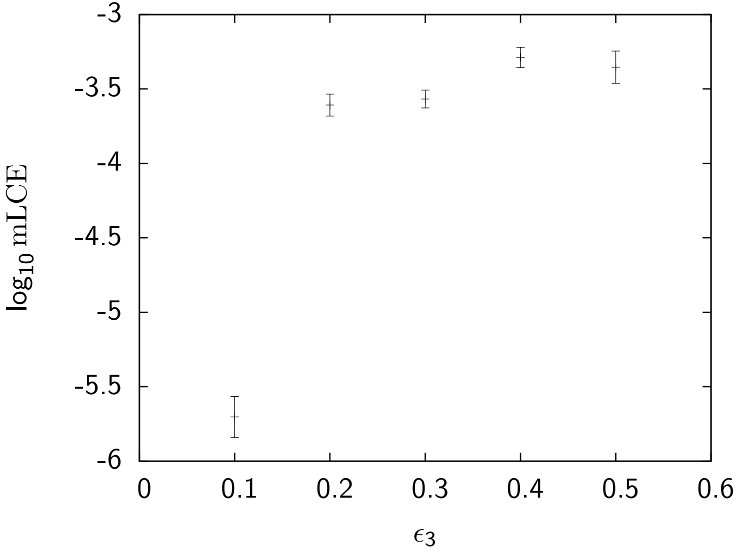
$$\mathrm{mLCE} = \lim_{\tau \to \infty} \frac{1}{\tau} \log \left[\frac{\Xi(\tau)}{\Xi(0)} \right] \tag{7}$$

- regular trajectory: $\Xi(\tau) \sim \tau$, $\mathrm{mLCE} = 0$
- chaotic trajectory: $\Xi(\tau) \sim e^{\lambda\tau}$, $\mathrm{mLCE} = \lambda$

Lyapunov exponents



Lyapunov exponents



Conclusion

It was shown by clear numerical examples using different methods that geodesic motion in the Johannsen-Psaltis spacetime corresponds in general to a non-integrable system.

Time for discussion

Any questions?